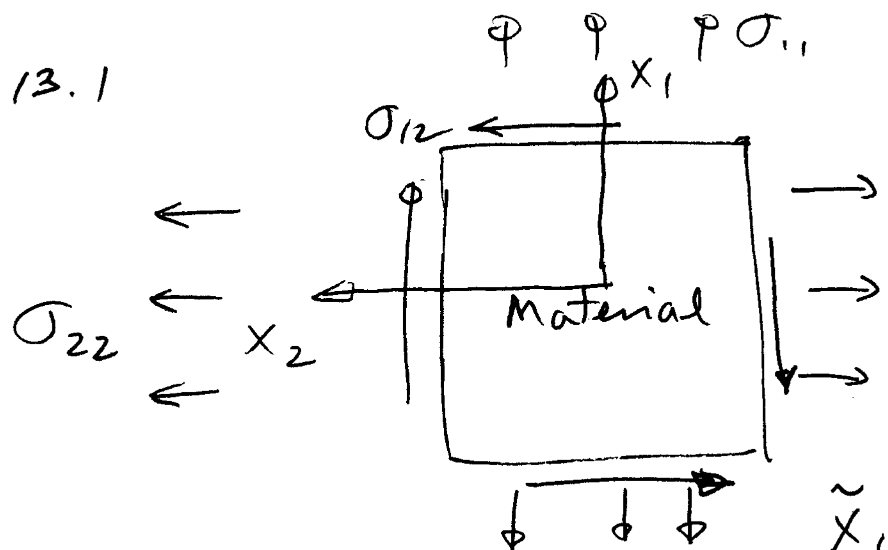
PAL
11/29/07

Unified Engineering Problem Set
Week 13 Fall, 2007

SOLUTIONS

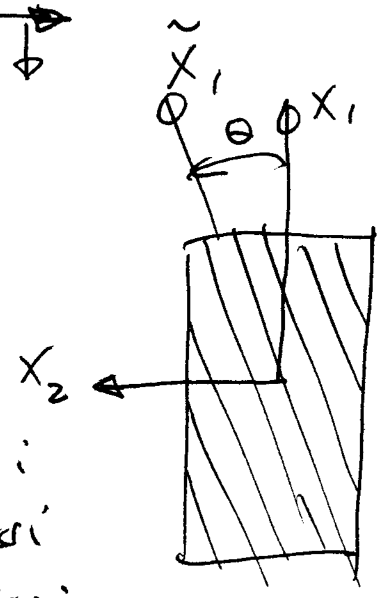
M13.1



$\sigma_{11} = 15 ksi$
$\sigma_{22} = -10 ksi$
$\sigma_{12} = -5 ksi$

(from M10.1)

$\tilde{\sigma}_{11} = 6.7 ksi$
$\tilde{\sigma}_{22} = -1.7 ksi$
$\tilde{\sigma}_{12} = -12.8 ksi$



(a) for plane stress, the principal stresses are the roots of the equation:

$$\sigma^2 - \sigma(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

We do this relative to the original loading axes. Using those stresses given:

$$\tau^2 - \tau(15\,ksi - 10\,ksi) + \left[(15\,ksi)(-10\,ksi) - (-5\,ksi)^2\right] = 0$$

$$\Rightarrow \tau^2 - (5\,ksi)\tau - 175\,(ksi)^2 = 0$$

Solve via the quadratic formula:

$$\text{roots: } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for } ax^2 + bx + c = 0$$

$$\Rightarrow \tau = \frac{-(-5\,ksi) \pm \sqrt{(-5\,ksi)^2 - 4(1)(-175)(ksi)^2}}{2(1)}$$

$$= \frac{5 \pm \sqrt{725}}{2}\,ksi$$

$$= \frac{5 \pm 26.9}{2}\,ksi$$

$$\Rightarrow \tau = 15.9\,ksi,\ -10.9\,ksi$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_I &= 15.9\,ksi \\ \sigma_{II} &= -10.9\,ksi \end{aligned}}$$

To find the associated directions, use the expression:

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)$$

$$\Rightarrow \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2(-5\,ksi)}{15\,ksi - (-10\,ksi)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(-\frac{10}{25}\right)$$

$$= \frac{1}{2}\tan^{-1}(-0.4)$$

$$\Rightarrow \theta_p = \frac{1}{2}(-21.8^\circ)$$

$$\Rightarrow \theta_p = -10.9^\circ \text{ for } \sigma_{I} \left(\begin{matrix}\text{check}\\ \text{manually}\end{matrix}\right)$$

with σ_{II} rotated 90° from that

so: $$\boxed{\begin{aligned}\theta_{p_I} &= -10.9^\circ \\ \theta_{p_{II}} &= -100.9^\circ = +79.1^\circ\end{aligned}}$$

Check the angles via the transformation equations for shear and that shear stress goes to zero (definition of principal axes and stresses):

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

for $\theta = -10.9^\circ$

$$\Rightarrow 0 \stackrel{?}{=} -(-0.189)(0.982)(15\,ksi) + (0.982)(-0.189)(-10\,ksi) + (0.964 - 0.036)(-5\,ksi)$$

$$0 \stackrel{?}{=} 2.78\,ksi + 1.86\,ksi - 4.64\,ksi$$

✓ (YES)

(same as for $+79.1^\circ$ with signs switched)

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(b) Maximum shear stress(es) occur along planes/directions that are at 45° to the principal axes:

So: direction of maximum shear stress:

$\theta_{p_I} + 45°$ from $x_1 = +35.9°$

$\theta_{p_{II}} + 45°$ from $x_1 = -55.9°$

$\boxed{\text{Direction of maximum shear} = +35.9°, -55.9°}$

The value of the maximum shear stress(es) in the $x_1 - x_2$ plane is:

$$\frac{\sigma_I - \sigma_{II}}{2} = \frac{15.9\,ksi - (-10.9\,ksi)}{2}$$

$\Rightarrow \boxed{\text{value of maximum shear stress} = 13.4\,ksi}$

(This takes on a sign of + and -)

NOTE:
This value can also be determined by using the stress transformation equation for σ_{12} and the directions of maximum shear relative to the original loading axes. So use:

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\sigma_{12}$$

with: $\sigma_{11} = 15\,ksi$
$\sigma_{22} = -10\,ksi$
$\sigma_{12} = -5\,ksi$

and $\theta = +35.9°, -55.9°$

There are two other maximum shear stresses out of the $x_1 - x_2$ plane. For the case of plane stress, the out-of-plane principal stress is zero ($\sigma_{III} = 0$).

So we have:

$$|\tau_{max}| = \left|\frac{\sigma_I - \sigma_{III}}{2}\right| = \frac{15.9 \text{ ksi}}{2} = \boxed{7.9 \text{ ksi}}$$

This is in a plane at 45° to the $x_1 - x_2$ plane rotated about the x_2-axis

$$|\tau_{max}| = \left|\frac{\sigma_{II} - \sigma_{III}}{2}\right| = \frac{10.9 \text{ ksi}}{2} = \boxed{5.5 \text{ ksi}}$$

This is in a plane at 45° to the $x_1 - x_2$ plane rotated about the x_1-axis

(c) The direction of the composite fiber axes (θ) do not change the base stress state and thus the principal stresses and maximum shear stresses $\boxed{\text{do not change.}}$

The principal directions and planes of the maximum stresses $\boxed{\text{stay the same}}$ relative to the loading axes $x_1 - x_2$. Only the fiber direction, θ, must be properly subtracted/added to get the direction relative to the fiber directions.

(d) Draw the Mohr's circle as specified in the instructions (see associated figure)

① Plot $\sigma_{11}, -\sigma_{12}$ (15ksi, +5ksi) as Point A

② Plot σ_{22}, σ_{21} (-10ksi, -5ksi) as Point B

③ Connect Ⓐ and Ⓑ

④ Draw circle of diameter of the line AB about the point where the line AB crosses the horizontal axis (denote this point as Ⓒ)

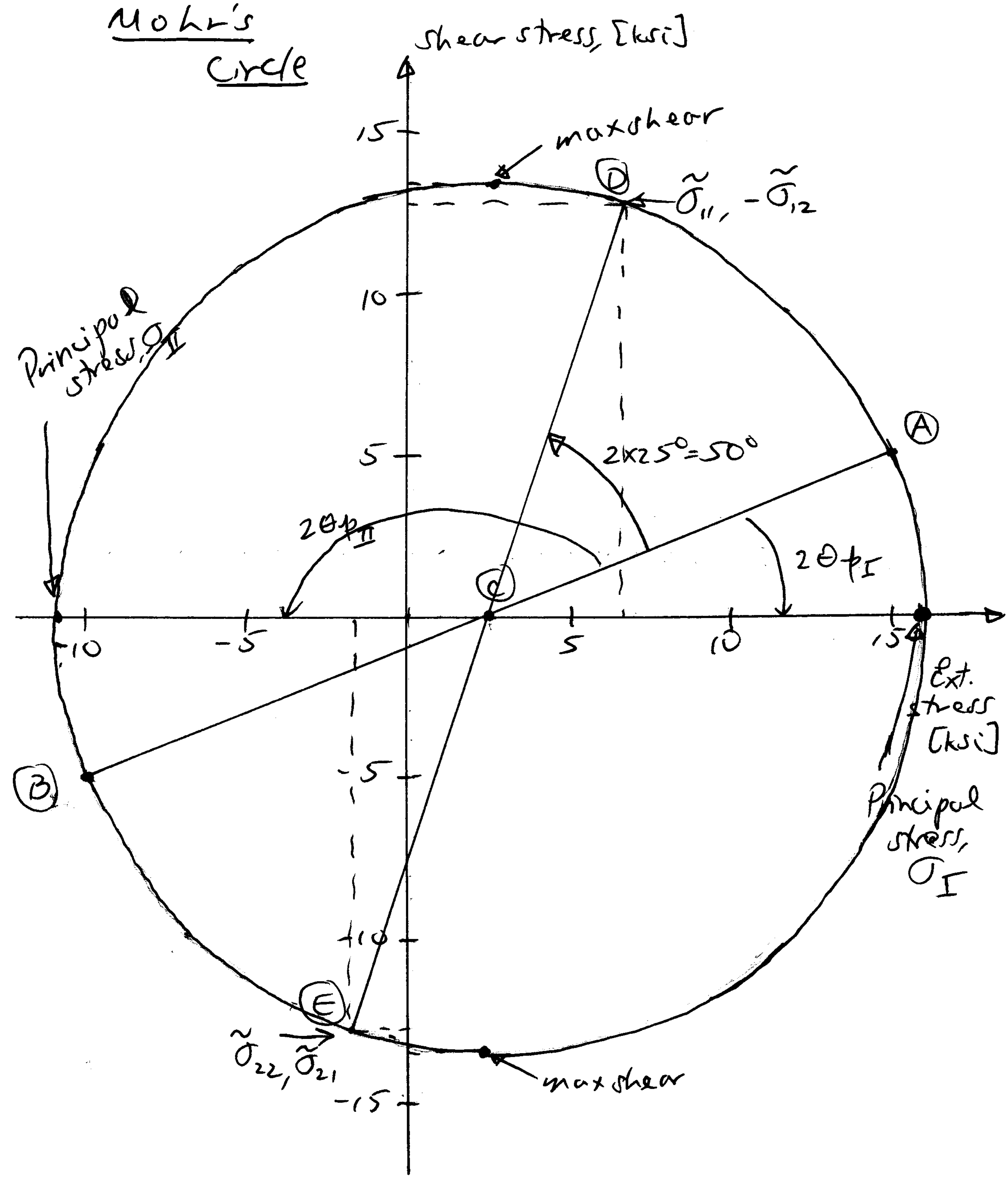
$$\text{point } Ⓒ = \text{midpoint} = \frac{\sigma_{11}+\sigma_{22}}{2} = \frac{15\text{ksi} - 10\text{ksi}}{2}$$

$$= 2.5\text{ksi}$$

→ Stresses in composite fiber axes

Rotate the diameter from the base plot by twice the angle of transformation ($2 \times 25° = 50°$). The intersection of the line with the circle at point Ⓓ are the values of $\tilde{\sigma}_{11}, -\tilde{\sigma}_{12}$. Read off as 6.7ksi, (-12.8ksi). As before.

The intersection at point Ⓔ gives the values of $\tilde{\sigma}_{22}, \tilde{\sigma}_{21}$. Read of as -1.7ksi, -12.8ksi. As before.


Mohr's Circle
shear stress, [ksi]
max shear
D
$\tilde{\sigma}_{11}, -\tilde{\sigma}_{12}$
Principal stress, σ_{II}
A
$2\times25^\circ=50^\circ$
$2\theta_{p_{II}}$
$2\theta_{p_I}$
C
Ext. stress [ksi]
B
Principal stress, σ_I
E
$\tilde{\sigma}_{22}, \tilde{\sigma}_{21}$
max shear
15
10
5
-5
-10
-15
-10
-5
5
10
15

→ Principal stresses and directions (2-D)

The intersection of the circle with the horizontal axis gives the two values of the principal stresses. By sight these are at the same values corresponding to the results of part (a): 15.9 ksi −10.9 ksi. One can be more formal by finding the circle diameter $\left(= 2\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}\right)$ and then adding and subtracting half of this from the midpoint value (point Ⓒ = 2.5 ksi)

Directions (angles) can be measured via a protractor and half the angle from line AB to the horizontal axis at the two points (2 directions)

→ Maximum shear stress

These are the upper and lower "reaches" of the circle along the vertical direction. It can be read off to be just about 13.5 ksi (and −13.5 ksi). This is very close to the value calculated in part (b) of ±13.4 ksi. Note that the value found via Mohr's circle is exactly the radius of the circle

The direction(s) of the maximum shear stress(es) are 90° on Mohr's circle from that of the principal stresses as these two associated lines are perpendicular. Divide by 2, since this is twice the rotation angle, and that is 45° added on to $\theta_{p_{I,II}}$ as in (b)

NOTE: Only the maximum shear stress in the $x_1 - x_2$ plane can be determined since Mohr's circle only allows rotation in the $x_1 - x_2$ plane.

M13.2 The following answers, as asked for in the problem statement, include a brief sentence on the primary functional requirement that needs to be met for each of the given cases. This includes the loads (e.g. tension, compression, shear, impact, cyclic, thermal, electrical) and five material properties that are most relevant to meeting this requirement. Note that the items listed are just ***some*** of the possible requirements, loads, and properties. (**NOTE:** Problem set answers will vary according to what the individual student indicates are the relevant loads and properties.)

(a) <u>Cable used in overhead cranes</u>: Must provide load-carrying capacity and resistance to environment for loads and items encountered in crane operations

Types of loads:
1. Tension (pulling)
2. Impact (jerks in operation)
3. Thermal (due to baseline temperature from environment)
4. Wear

Material properties:
1. Strength - High
2. Abrasion and wear - High
3. Modulus - High
4. Corrosion - High
5. Price - Low

(b) <u>Components of a truss used in a bridge</u>: Must provide load-carrying capacity for loads that a bridge undergoes.

Types of loads:
1. & 2. Tension and Compression (depending on design)
3. Assembly
4. Environmental (Thermal, Corrosive aspects)

Material properties:
1. Modulus - High
2. Corrosion/Longevity - High
3. Strength - Medium
4. Fabrication & Joining - High
5. Price - Low

(c) <u>Kitchen countertop</u>: Must provide an "aesthetic" work surface for a kitchen.

Types of loads:
1. Impact
2. Compression
3. Thermal

Material properties:
1. Price - Low
2. Availability - High
3. Hardness - Medium
4. Appearance - High
5. Finishing - High

(d) Combustion chamber lining of a jet engine: Must provide contained thermal environment for operation of engine.

Types of loads:
1. Impact
2. Thermal
3. Aerodynamic/Pressure
4. Wear

Material properties:
1. Thermal - High
2. Modulus - High
3. Impact - High
4. Corrosion - High
5. Density - Low

(e) Components of a space truss: Must provide load-carrying capacity for loads that a space truss undergoes.

Types of loads:
1. Impact (docking)
2. Thermal (solar)
3. & 4. Tension and Compression (depending on design)
5. Cyclic

Material properties:
1. Thermal - High
2. Density - Low
3. Modulus - High
4. Joining - Medium
5. Longevity - High

(f) Leading edge of the space shuttle: Provide defined aerodynamic surface for shuttle wing.

Types of loads:
1. Thermal
2. Aerodynamic
3. Impact
4. Cyclic

Material properties:
1. Thermal - High
2. Density - Low
3. Oxidation Resistance - High
4. Impact - High
5. Strength - Medium

M13.3 Structural member is of a given length with a square box cross-section and the member must carry a constant load, in tension or compression, of no greater magnitude than P. The box has an inner wall length of a.

Representation:

δ = bar deformation

(a) List the constants: P, L, a

Requirement: Carry load of a magnitude no more than P

Needs: Deform as little as possible and weigh as little as possible

Design variables: structural member wall thickness (t) and material used

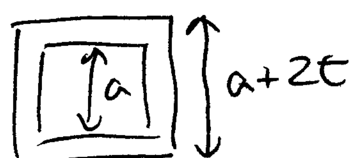
→ List items to be considered for minimization, etc.:

mass/weight (m)
deformation (δ)
cost (c)

→ List key equations:

stress-strain: $\sigma = E\epsilon$ (1)
strain-displacement: $\epsilon = \delta/L$ (2)
stress-load: $\sigma = P/A$ (3)
area-thickness

for a box with walls of thickness t:



the outer walls are of length $a+2t$. Thus, subtracting the ~~inner~~ box area from the outer box area gives the area:

$$(a+2t)^2 - a^2 = A$$

$$\Rightarrow A = a^2 + 4at + 4t^2 - a^2$$

$$A = 4at + 4at^2$$

$$\text{if } a \gg t \Rightarrow A \simeq 4at$$

This is the same as looking at each inner wall x thickness = at and multiplying by 4 (there are four of them).

So: $A = 4at$ (4)

mass (weight) - density: $M = \rho A L$ (5)

Cost: $\left(\frac{\text{cost}}{\text{weight}}\right)(\text{weight}) = \text{Cost}$ (6)

→ List other variables, parameters:

E = modulus
t = thickness
$\updownarrow$
A = Area
ρ = density

The figures of merit are based on the overall items to be considered and expressing these in terms of geometrical and material parameters/properties.

→ first consider the deformation:

- from (2): $\delta = \epsilon L$
- use (1) to give: $\epsilon = \sigma / E$
  and thus: $\delta = \frac{\sigma L}{E}$

- Now use (3) in this:
$$\delta = \frac{PL}{EA}$$

- Finally use (4) to get this in terms of load, length, wall thickness and modulus:

$$\boxed{\delta = \frac{PL}{4Eat}}$$

First
Figure of Merit
(*1)

→ Now consider mass/weight:

- from (5): weight = ρAL
  ↑ weight density

- Use (4) to get in terms of load, length and wall thickness along with inner wall length:

$$\text{weight} = \rho 4atL$$

$$\Rightarrow \boxed{\text{weight} = 4\rho atL}$$

Second
Figure of Merit
(*2)

→ Finally consider cost:

- from (6):

$$\text{cost} = c\,(\text{weight})$$

↑ cost/weight

- Using the second Figure of Merit gets this in terms of key parameters:

$$\boxed{\text{Cost} = 4\, C \rho a t L}$$

Third
Figure of Merit
(* 3)

(b) We have three equations that allow us to explore the possibilities in terms of the key items (deformation, weight, cost).

However, separately considering any one item and its minimization (as specified in this case) is generally insufficient. For example, one can decrease deformation by continuing to increase member wall thickness. The key is to consider the ability of any choice for other fixed considerations. This leads to considering **tradeoffs** !

If one considers the ability to provide a specified minimum deformation (call it δ_o) and first consider mass (weight), the first and second figures of merit can be combined for this consideration:

from (*1): $\delta_o = \dfrac{PL}{4Eat}$

$\Rightarrow t = \dfrac{PL}{4Ea\delta_o}$

Place this in (*2):

$$\text{weight} = 4\rho a \left(\frac{PL}{4Ea\delta_o}\right)L$$

$$\Rightarrow \text{Weight} = \frac{\rho PL^2}{E\delta_o}$$

Here, $\frac{PL^2}{\delta_o}$ is a constant, so we assess the possibilities via the factor ρ/E

Material	ρ/E $[\text{lb}/\text{in}^{-3}]/[\frac{10^6\,\text{lb}}{\text{in}^2}] = [\frac{1}{10^6\,\text{in}}]$
Aluminum	0.0096
Carbon fiber Composite	0.0022
$\boxed{\text{Silicon Carbide}}$	0.0018
Steel	0.0098
Titanium	0.0100
Wood	0.0122

best choice to minimize mass/weight for a given deformation

→ Now consider cost by using (* 3)

$$\Rightarrow \text{Cost} = 4C\rho a \left(\frac{PL}{4Ea\delta_o}\right)L$$

giving .... $\text{Cost} = \frac{CPL^2\rho}{E\delta_o}$

Here, $\frac{PL^2}{\delta_o}$ is, again, a constant. So we assess the possibilities via the factor $C\rho/E$

material	$\frac{C\rho}{E}$ $\left[\frac{\$}{10^6 \text{ in}\cdot\text{lb}}\right]$
Aluminum	0.060
Carbon fiber Composite	0.187
Silicon Carbide	0.263
Steel	0.013 * ← a close second
Titanium	0.251
$\boxed{\text{Wood}}$	0.012 ← best choice to minimize cost for a given displacement

→ Finally, think about minimizing deformation for a fixed cross-section.

using (*1):

$$\delta = \frac{PL}{4Eat}$$

Here, $\frac{PL}{4at}$ is given, so we assess the possibilities via the factor $1/E$

Material	$1/E$ $\left[\frac{in^2}{10^6 lb}\right]$
Aluminum	0.095
Carbon fiber Composite	0.041
$\boxed{\text{Silicon Carbide}}$	0.017
Steel	0.034
Titanium	0.063
Wood	0.552

best choice to minimize deformation (for a given area, and thus weight)

Final note: There is no final answer without further clarification of the objectives and the relative value of the various aspects/criteria. It depends on decisions with regard to the tradeoffs.